Noncommutative 2-Dimensional Models of Gravity

M. Burić[1], J. Madore[2]

[1]Faculty of Physics, P.O. Box 368
11001 Belgrade

[2]Laboratoire de Physique Théorique
Université de Paris-Sud, Bâtiment 211, F-91405 Orsay

Abstract

A review is given of some 2-dimensional metrics for which noncommutative versions have been found. They serve partially to illustrate a noncommutative extension of the moving-frame formalism. All of these models suggest that there is an intimate relation between noncommutative geometry on the one hand and classical gravity on the other.

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1 Introduction

There is a very simple argument due to Pauli that the quantum effects of a gravitational field will in general lead to an uncertainty in the measurement of space coordinates. It is based on the observation that two ‘points’ on a quantized curved manifold can never be considered as having a purely space-like separation. If indeed they had so in the limit for infinite values of the Planck mass, then at finite values they would acquire for ‘short time intervals’ a time-like separation because of the fluctuations of the light cone. Since the ‘points’ are in fact a set of four coordinates, that is scalar fields, they would not then commute as operators. This effect could be considered important at least at distances of the order of Planck length, and perhaps greater. This is one motivation to study noncommutative geometry. A second motivation, which is the one we consider ours, is the fact that it is possible to study noncommutative differential geometry, and there is no reason to assume that even classically coordinates commute at all length scales. One can consider for example coordinates as order parameters as in solid-state physics and suppose that singularities in the gravitational field become analogs of core regions; one must go beyond the classical approximation to describe them. A straightforward and conservative way to do this is to represent them by operators. The space-time manifold is thus replaced with an algebra $\mathcal{A}$ (noncommutative ‘space’), generated by a set of noncommutative ‘coordinates’ x^i. We think of x^i as linear operators on some vector space, and therefore we assume that the multiplication in the algebra is associative. The essential element which allows us to interpret a noncommutative algebra as a space-time is the possibility [1, 2] to introduce a differential structure on the former.

We use a noncommutative version of Cartan’s frame formalism [3]; the differential structure has been also studied from other points of view [2, 4, 5]. In order to develop some intuition in the complete absence of experimental evidence, one is obliged to consider examples. Several of these have been found. We shall introduce as illustrations the quantum plane, the 2-dimensional de Sitter space, the 2-dimensional Rindler space and the ‘fuzzy donut’. Their simplicity will allow us to bypass the general formalism and will permit a more intuitive presentation. A series of models in all dimensions has been found [6], as well as some models in dimensions two [3, 7] and four [8, 9]. We shall argue that the moving frame formalism is in this respect a natural way to implement gravity. It enables one to introduce a sort of correspondence principle as a guide of how to construct the frame from its commutative limit. Some of our examples have been considered elsewhere; we believe the Rindler and the donut examples to be new. We discuss their properties in more detail in the last sections and therefore on a less introductory level. The parameter $\mathchar'26\mkern-9mu k$ with the dimensions of length squared is introduced in Section 4 to facilitate the discussion of the commutative limit; a mass parameter μ as well as Newton’s constant, are also introduced.

2 Differential calculi

In the following sections we shall use some known examples to introduce the reader to the more elementary aspects of the noncommutative extension of the de Rham differential calculus. In ordinary geometry a topological manifold can have more than one differential structure. In the noncommutative case this is generically so: many non-equivalent differential structures on a given algebra exist. This means that there are several ‘geometries’ which one can associate to the underlying noncommutative space. Once the differential calculus is chosen, some more or less obvious assumptions

as hermiticity and bilinearity fix the linear connection almost uniquely. This is to be contrasted with the commutative case, where for each differential structure linear connection can be chosen almost arbitrarily. The choice therefore of a differential is one of the more important steps in the 'quantization'. We stress that we do not think that all noncommutative geometries are suitable as noncommutative models of space-time, any more so than in the commutative limit.

The outline of the presentation is the following. In Section 3 we shall introduce derivations as the noncommutative version of vector fields and in Section 4 we shall consider more particularly inner derivations and introduce momentum generators. In Sections 5,6 and 7 we discuss the nonequivalence of the calculi induced by different frames, the frame rotations and the noncommutative limit. In Section 8 we show that the momenta close under the commutator to generate a quadratic algebra. Implicit in our calculations is the assumption that this algebra can be at least formally identified with the original algebra generated by the 'coordinates'. Although not explicitly mentioned, this algebra is present in most of the examples; the Section 6 presents what could be considered a counterexample with an algebra which is not quadratic. We conclude by introducing connection, metric and curvature in the noncommutative case in Section 9 and discussing their relations in Section 10.

3 Quantum plane: differentials

We introduce here the quantum plane to illustrate the basic features of noncommutative geometry; more details can be found for example in the books mentioned above. A noncommutative 'space' is an associative $*$-algebra Ą generated by a set of hermitean 'coordinates' x^i which in some limit tend to the (real) coordinates $\tilde{x}^i$ of a manifold; the latter we identify as the classical limit of the geometry. That is, in the classical or weak-field limit we impose the condition

$$x^i \to Z^{-1}\tilde{x}^i. \tag{3.1}$$

The Z could be perhaps singular; in all examples considered here we can choose $Z = 1$. Elements of Ą will be denoted by x^i, f, g, p_a, and so forth. In general the coordinates satisfy a set of commutation relations. We shall consider the algebra as a formal algebra and not attempt to represent it as an algebra of operators.

The simplest relation which can be used to define the algebra is

$$\big[x^i, x^j\big] = iJ^{ij}, \tag{3.2}$$

where J^{ij} are real numbers defining a canonical or symplectic structure. Often in the literature the notation θ^{ij} is used instead of J^{ij}. Another associative algebra is defined using the commutation relations

$$\big[x^i, x^j\big] = iC^{ij}{}_k x^k. \tag{3.3}$$

For simplicity we shall assume that the center of the algebra, the set of elements which commute with all generators, consists of complex multiples of the identity. The third important special case is a quantum space, defined by a homogeneous quadratic relation:

$$\big[x^i, x^j\big] = iC^{ij}{}_{kl} x^k x^l. \tag{3.4}$$

A combination of these three commutation relations will be satisfied by a set of generators p_a of $\mathcal{A}$ which we shall refer to as momenta; the commutation relations obeyed by the 'coordinates' in general are not even necessarily polynomial.

In ordinary geometry a vector field can be defined as a derivation of the algebra of smooth functions. This definition can be used also when the algebra is noncommutative. A derivation, we recall, is a linear map $f \mapsto Xf$ which satisfies the Leibniz rule, $X(fg) = (Xf)g + fXg$. Derivations will be denoted by X, Y, e_a and so forth and the set of all derivations by $\mathrm{Der}(\mathcal{A})$. A simple example is the algebra of 2×2 complex matrices M_2 with (redundant) generators the Pauli matrices. The algebra is of dimension four, the center is of dimension one and $\mathrm{Der}(M_2)$ is of dimension three with basis consisting of three derivations $e_a = \mathrm{ad}\,\sigma_a$:

$$e_a f = [\sigma_a, f]. \tag{3.5}$$

We notice that the Leibniz rule is here the Jacobi identity. We see also that the left multiplication $\sigma_a e_b$ of the derivation e_b by the generator σ_a no longer satisfies the Jacobi identity: it is not a derivation. The vector space $\mathrm{Der}(M_2)$ is not a left M_2 module. This property is generic. If X is a derivation of an algebra $\mathcal{A}$ and h an element of $\mathcal{A}$, then hX is not necessarily a derivation:

$$hX(fg) = h(Xf)g + hfXg \neq h(Xf)g + fhXg \tag{3.6}$$

if $hf \neq fh$. Notice that the derivations of M_2 are inner: they were defined as a commutator with an element of the algebra. It is a simple theorem that all derivations of a complete matrix algebra are inner. On the other hand, the derivations on an algebra of functions are not inner; they are known as outer.

Although derivations do not form a left module, one can introduce associated elements known as differential forms which form a bimodule; they can be multiplied from the left and from the right. We shall therefore express as much as possible physical quantities using the latter. We define here a 1-form ω is a linear map $\omega : \mathrm{Der}(\mathcal{A}) \to \mathcal{A}$. The set of 1-forms $\Omega^1(\mathcal{A})$ has a bimodule structure, that is, if ω is a 1-form, $f\omega$ and ωf are also 1-forms. The elements of $\Omega^1(\mathcal{A})$ will be typically denoted by ω, θ, ξ, η.

The important step is the definition of a differential d; it is a linear map from functions to 1-forms, $d : \mathcal{A} \to \Omega^1(\mathcal{A})$ which obeys the Leibniz rule. In general $fdg \neq dgf$ but we shall introduce later special forms θ^a which commute with the algebra. The exterior product $\xi\eta$ of two 1-forms ξ and η is a 2-form. There is no reason to assume the exterior product antisymmetric. We mention also that one can deduce the structure of the algebra of all forms from that 'of the module of 1-forms. The map d can be extended to all forms if one require that $d^2 = 0$. We should stress that in general one can associate many differential calculi to a given algebra.

To illustrate these notions, we construct a differential for the canonical structure (3.2): $\left[x^i, x^j\right] = iJ^{ij}$. From the Leibniz rule it follows that the differential of the unit element must vanish. Therefore the differential must satisfy the constraint

$$0 = d\left[x^i, x^j\right] = dx^i x^j + x^i dx^j - dx^j x^i - x^j dx^i. \tag{3.7}$$

A possible but not unique solution to this equation is

$$dx^i x^j - x^j dx^i = \left[dx^i, x^j\right] = 0. \tag{3.8}$$

Furthermore, the relations of the algebra imply that

$$0 = d\left[dx^i, x^j\right] = (d^2 x^i)x^j - dx^i dx^j - dx^j dx^i + x^j (d^2 x^i), \tag{3.9}$$

that is, the differentials anticommute as if they were defined on a manifold,

$$dx^i dx^j = -dx^j dx^i. \tag{3.10}$$

For the Lie algebra (3.3) however, we see that we could not have imposed the condition $\left[dx^i, x^j\right] = 0$ as it is inconsistent with the relation $\left[x^i, x^j\right] = iC^{ij}{}_k x^k$. It would imply $C^{ij}{}_k dx^k = 0$.

The first example we discuss in detail is the quantum plane. It has two generators x and y related by

$$xy = qyx, \tag{3.11}$$

where q is a constant which we shall assume not to be a root of unity. For example, two derivations e_1, e_2 can be defined by the formulae

$$\begin{aligned} e_1 x &= x, & e_2 x &= 0, \\ e_1 y &= 0, & e_2 y &= y. \end{aligned} \tag{3.12}$$

These would necessarily be outer derivations. There are other possibilities. Let e_a be defined by

$$\begin{aligned} e_1 x &= \frac{q^2}{q^2+1} x^{-1} y^2, & e_1 y &= \frac{q^4}{(q^2+1)} x^{-2} y^3, \\ e_2 x &= 0, & e_2 y &= -\frac{q^2}{(q^2+1)} x^{-2} y. \end{aligned} \tag{3.13}$$

These derivations are, as we shall see, inner.

A differential must satisfy the constraint

$$d(xy - qyx) = dx\, y + x\, dy - q\, dy\, x - qy\, dx = (dx\, y - q\, dy\, x) + (x\, dy - qy\, dx) = 0. \tag{3.14}$$

This we can satisfy, for example, by setting

$$dx\, y - qy\, dx = 0, \quad q dy\, x - x\, dy = 0, \tag{3.15}$$

which defines the commutation rules of dx and y, and dy and x. In order to complete the definition one must add the rules for dx and x and dy and y. For example,

$$\begin{aligned} dxy &= qdyx, & x\, dy &= q\, dy\, x, \\ x\, dx &= q\, dx\, x, & qy\, dy &= dy\, y. \end{aligned} \tag{3.16}$$

Applying d once more, one thus obtains the exterior product

$$(dx)^2 = 0, \quad (dy)^2 = 0, \quad dx\, dy = -q\, dy\, dx. \tag{3.17}$$

For reasons [10] which do not concern us here (deformed symmetries), one prefers another differential calculus constructed by setting instead of (3.15)

$$dx\, y - qy\, dx = (1 - q^2) dx\, y, \quad q\, dyx - x\, dy = (1 - q^2) dx\, y. \tag{3.18}$$

The full set of relations for the 1-differential forms would be in this case

$$\begin{aligned} q\, dx\, y &= y\, dx, & x\, dy &= q\, dyx + (q^2 - 1) dx\, y, \\ x\, dx &= q^2 dx\, x, & y\, dy &= q^2 dy\, y, \end{aligned} \tag{3.19}$$

In this case the exterior product is given by

$$(dx)^2 = 0, \quad (dy)^2 = 0, \quad q\, dx\, dy = -dy\, dx. \tag{3.20}$$

We shall see that it is based on the inner derivations (3.13) defined above. The relation between the unusual structure of these derivations and the deformed symmetries is not completely understood.

4 De Sitter: frames

As we have seen, there is a variety of possibilities to define a differential. One problem is how to determine or at least restrict it by imposing some physical requirements. We shall use here a modification of the moving frame formalism and show that so defined differential calculi over an algebra admit essentially a unique metric and linear connection. We shall fix therefore the differential calculus by requiring that the metric have the desired classical limit. The idea is to define an analogue of a parallelizable manifold, which therefore has a globally defined frame. The frame is defined either as a set of vector fields e_a or as a set of 1-forms θ^a dual to them. The metric components with respect to the frame are then constant.

We choose a set of n derivations e_a which we assume to be inner generated by 'momenta' p_a:

$$e_a f = [p_a, f]. \tag{4.1}$$

We suppose that the momenta generate also the whole algebra $\mathcal{A}$. Since the center is trivial, this means that an element which commutes with all momenta must be a complex number. An alternative way is to use the 1-forms θ^a dual to e_a such that relation

$$\theta^a(e_b) = \delta^a_b \tag{4.2}$$

holds. To define the left hand side of this equation we define first the differential, exactly as in the classical case, by the condition

$$df(e_a) = e_a f. \tag{4.3}$$

The left and right multiplication by elements of the algebra $\mathcal{A}$ are defined by

$$f dg = f e_a g \theta^a, \qquad dg f = e_a g f \theta^a. \tag{4.4}$$

Since every 1-form can be written as sum of such terms the definition is complete. In particular, since

$$f\theta^a(e_b) = f\delta^a_b = (\theta^a f)(e_b), \tag{4.5}$$

we conclude that the frame necessarily commutes with all the elements of the algebra $\mathcal{A}$; this is a characteristic feature. If one does not insist on using differential calculi defined by inner derivations this condition can be generalized to include frames which commute only modulo an algebra morphism. For a recent discussion of this possibility we refer to [11].

In the case of the algebra M_2 considered above, the module of 1-forms is generated by three elements $d\sigma_a$ defined as the maps

$$d\sigma_a(e_b) = e_b\sigma_a = [\sigma_b, \sigma_a]. \tag{4.6}$$

The maps $\sigma_c d\sigma_a$ and $d\sigma_a\sigma_c$ are defined respectively as

$$\sigma_c d\sigma_a(e_b) = \sigma_c[\sigma_b, \sigma_a], \qquad d\sigma_a\sigma_c(e_b) = [\sigma_b, \sigma_a]\sigma_c. \tag{4.7}$$

Obviously, $\sigma_c d\sigma_a \neq d\sigma_a\sigma_c$.

The 1-form θ defined as

$$\theta = -p_a\theta^a \tag{4.8}$$

can be considered as an analog of the Dirac operator in ordinary geometry. It implements the action of the exterior derivative on elements of the algebra. That is

$$df = -\left[\theta, f\right] = \left[p_a\theta^a, f\right] = \left[p_a, f\right]\theta^a. \tag{4.9}$$

The differential is real if $(df)^* = df^*$. This is assured if the derivations e_a are real: $e_a f^* = (e_a f)^*$, which is the case if the momenta p_a are antihermitean. From the definitions one has $\theta^{a*} = \theta^a$, $\theta^* = -\theta$. Furthermore, $(f\xi)^* = \xi^* f^*$, $(\xi f)^* = f^*\xi^*$, and $(\xi\eta)^* = -\eta^*\xi^*$. Note that whereas the product of two hermitean elements is hermitean only if they commute, the product of two hermitean 1-forms is hermitean only if they anticommute.

Consider once more the quantum plane introduced in the previous section. The momenta p_a can be defined as

$$p_1 = \frac{q}{q-1}\,y, \quad p_2 = \frac{q}{q-1}\,x. \tag{4.10}$$

From these expressions one easily finds the relations

$$\begin{aligned} e_1 x &= -xy, & e_2 x &= 0,\\ e_1 y &= 0, & e_2 y &= xy. \end{aligned} \tag{4.11}$$

Using the definition $df = e_a(f)\theta^a$, one obtains for θ^1 and θ^2,

$$\theta^1 = -y^{-1}x^{-1}dx, \quad \theta^2 = y^{-1}x^{-1}dy. \tag{4.12}$$

From (4.5) the module structure (3.16) can be reconstructed. The momenta p_a satisfy the quadratic relation

$$p_2 p_1 = q p_1 p_2. \tag{4.13}$$

The second differential calculus [10] on the quantum plane also has a frame. The corresponding momentum generators are

$$p_1 = \frac{1}{q^4-1}\,x^{-2}y^2, \quad p_2 = \frac{1}{q^4-1}\,x^{-2}. \tag{4.14}$$

They satisfy

$$p_1 p_2 = q^4 p_2 p_1. \tag{4.15}$$

Note that the momenta are singular in the limit $q \to 1$. In quantum mechanics the relation between the differential and the momentum p is given by

$$\frac{\partial f}{\partial x} = \frac{i}{\hbar}\,[p, f], \tag{4.16}$$

whereas it is given here by the expression (4.1). The singularity of the classical limit $\hbar \to 0$ has been included in the definition of the momentum.

The implementation of the differential structure as we have given is just as arbitrary as before since it amounts to a choice of the momenta. In some cases, the construction of the frame is not difficult. In the example (3.2) one can choose the differential such that $\left[x^i, dx^j\right] = 0$; a frame is $\theta^a = \delta^a_i dx^i$ since dx^i commute with all elements of the algebra. The most general form is $\theta^a = \Lambda^a_i dx^i$ with Λ^a_i real numbers. The duality relations give the momenta

$$\delta^a_b = \theta^a(e_b) = \delta^a_i dx^i(e_b) = \delta^a_i e_b(x^i) = \delta^a_i [p_b, x^i], \tag{4.17}$$

that is,

$$p_b = -i\delta_b^k J_{ki}^{-1} x^i. \tag{4.18}$$

In order to discuss noncommutative limit, (3.2) should in fact be rewritten as

$$[x^i, x^j] = i\kappa\!\!\!{}^- J^{ij}, \tag{4.19}$$

introducing the parameter $\bar{k}$ to describe the fundamental area scale on which noncommutativity becomes important. The $\bar{k}$ is presumably of order of the Planck area $G\hbar$; the commutative limit is defined by $\bar{k} \to 0$. The momenta read then

$$p_a = \frac{1}{i\bar{k}} J_{ai}^{-1} x^i, \tag{4.20}$$

and they are singular in the limit $\bar{k} \to 0$.

Since the frame is given by $\theta^a = \delta_i^a dx^i$, this space can be naturally thought of as the noncommutative generalization of flat space. The momenta are linear in the coordinates and hence

$$[p_a, p_b] = \frac{1}{(i\bar{k})^2} J_{ai}^{-1} J_{bj}^{-1} \left[x^i, x^j\right] = K_{ab}, \qquad K_{ab} = -\frac{1}{i\bar{k}} J_{ab}^{-1}. \tag{4.21}$$

In general only by explicit construction can one show that the frame exists. In the case of the Lie algebra (3.3), for example, one sees that the 1-forms dx^i do not define a frame because they do not commute with the algebra. In the example of M_2 with Pauli matrices as momenta, the frame which is the solution to the equation (4.2) is seen to be

$$\theta^a = \tfrac{1}{4}\sigma_b \sigma^a d\sigma^b. \tag{4.22}$$

This construction can be repeated [12] for the algebra M_n of $n \times n$ complex matrices.

As the main example of this section we consider the algebra generated by two hermitean elements x and y related by

$$[x, y] = -2i\bar{k}\mu y. \tag{4.23}$$

This is related to the Jordanian deformation [13] of $\mathbb{R}^2$ with deformation parameter $h = i\bar{k}\mu^2$. The μ is the gravitational mass scale; the associated length $G\mu$ vanishes with $\bar{k}$. To find the frame, we rewrite this as follows

$$(x + i\bar{k}\mu)y = y(x - i\bar{k}\mu). \tag{4.24}$$

The differential must satisfy

$$dx\, y + (x + i\bar{k}\mu)\, dy = dy\, (x - i\bar{k}\mu) + y\, dx. \tag{4.25}$$

We shall impose separately the conditions

$$dx\, y = y\, dx, \qquad (x + i\bar{k}\mu)\, dy = dy\, (x - i\bar{k}\mu). \tag{4.26}$$

The first of these relations suggests that dx can be taken as a frame element, in fact $f(y)dx$ as well as dx. We set $\theta^1 = f(y)dx$. Rewriting (4.26) as

$$(x + i\bar{k}\mu)yy^{-1}\, dy = yy^{-1}\, dy\, y^{-1}y(x - i\bar{k}\mu), \tag{4.27}$$

we see that we can take $\theta^2 = -(\mu y)^{-1} dy$. We assume that θ^a commute with x and y. The duality relations (4.2) determine $f(y)$. They read

$$\begin{aligned} f(y)[p_1, x] &= 1, & f(y)[p_2, x] &= 0, \\ (\mu y)^{-1}[p_1, y] &= 0, & (\mu y)^{-1}[p_2, y] &= -1, \end{aligned} \tag{4.28}$$

and reduce to (4.26) for

$$p_2 = \frac{1}{2i\hbar}\, x, \qquad p_1 = \frac{1}{2i\hbar}\, y, \qquad f(y) = (\mu y)^{-1}. \tag{4.29}$$

The frame therefore is given by

$$\theta^1 = (\mu y)^{-1} dx, \quad \theta^2 = -(\mu y)^{-1} dy. \tag{4.30}$$

The corresponding calculus is the covariant one [13]. The momenta are proportional to the coordinates so their commutation relation is:

$$[p_1, p_2] = \mu p_1. \tag{4.31}$$

A short calculation shows that the frame elements anticommute. The line element

$$ds^2 = \pm(\tilde\theta^1)^2 \pm (\tilde\theta^2)^2 \tag{4.32}$$

of the commutative limit is that of the Lobachevski plane or of (anti) de Sitter space depending on the choice of signature.

5 Fuzzy sphere: generalized frames

We now present an example of a calculus which possesses a frame but not one which is dual to a set of derivations. It is a 2-dim calculus which resembles a noncommutative version of the de Rham calculus over the 2-sphere. As such it can of course have no commutative limit. Consider an algebra with two generators χ, ϕ and introduce a frame

$$\theta^1 = d\chi, \qquad \theta^2 = \sin\chi\, d\phi. \tag{5.1}$$

The parameter μ is the inverse radius of the sphere and we set $\epsilon = \mu^2\hbar$. As usual we denote $[\chi, \phi] = i\epsilon J^{12}$ and assume that the frame elements commute with the generators χ, ϕ. This gives immediately the relations

$$[\chi, d\phi] = 0, \quad [\chi, d\chi] = 0, \quad [\phi, d\chi] = 0, \tag{5.2}$$

which imply that

$$(\theta^1)^2 = 0, \quad \theta^1\theta^2 = -\theta^2\theta^1, \tag{5.3}$$

as well as

$$dJ^{12} = 0. \tag{5.4}$$

We set $J^{12} = 1$. The remaining relations yield the identity

$$[\phi, d\phi] = i\epsilon \cot\chi\, d\phi. \tag{5.5}$$

We take the differential to obtain

$$(\theta^2)^2 = -i\epsilon \frac{1}{2\sin\chi}\, \theta^1\theta^2. \tag{5.6}$$

We have defined a differential calculus with the differential

$$d\theta^1 = 0, \quad d\theta^2 = \cot\chi\, \theta^1\theta^2, \tag{5.7}$$

but as we shall now show not one which is dual to a set of inner derivations.

The relations

$$\begin{aligned} [p_1, \phi] &= 0, & [p_1, \chi] &= 1, \\ [p_2, \phi] &= \frac{1}{\sin\chi}, & [p_2, \chi] &= 0 \end{aligned} \tag{5.8}$$

define the duality between momenta and coordinate generators. These can be solved to yield

$$i\epsilon p_1 = -\phi, \qquad i\epsilon p_2 = G(\chi), \tag{5.9}$$

with the function $G(\chi)$ defined by

$$\frac{dG}{d\chi} = \frac{1}{\sin\chi}, \quad G(\chi) = \log\tan\frac{\chi}{2}. \tag{5.10}$$

Using these momenta we define a 'Dirac operator'

$$\theta = -p_a\theta^a. \tag{5.11}$$

From (5.9) we find that θ is given by

$$\theta = \frac{1}{i\epsilon}\phi d\chi - \frac{1}{i\epsilon}G(\chi)\sin\chi d\phi. \tag{5.12}$$

We have then

$$d\theta + \theta^2 = -\frac{1}{i\epsilon}\frac{1}{2\sin\chi}(4 + 2G\cos\chi + G^2)\theta^1\theta^2. \tag{5.13}$$

As we shall see later (Equation (8.3)) the commutator of f with (5.13) defines the second differential d^2f of an element f. Therefore it must commute with all elements of the algebra; this is obviously not the case with (5.13). Thus we see that the differential defined by the set of momenta (5.9) is inconsistent.

A solution [14, 3] to this problem is to consider only 2-forms modulo the image of d^2. This would result in a consistent differential calculus but with 2-forms depending only on ϕ, with perhaps special functions of χ. One should not of course be too attached to the condition $d^2 = 0$. A non-vanishing value for this operator could be interpreted as some sort of 'micro-curvature'. In a subsequent article [15] the authors will examine the relation between this 'micro-curvature' and ordinary curvature using the WKB approximation.

6 Rindler space: frame rotations

In the last example of the previous section we saw that typically there is little freedom in finding the solution to duality and consistency equations. This is due to the relations among the momentum generators which we shall derive in Section 8. We shall see there that this relation is at most quadratic. As an example of a frame for which such a noncommutative extension does not exist we consider the 2-dim Rindler frame which is defined in one-half of 2-dim Minkowski space. We shall use this example to illustrate the fact that not all moving frames are suitable for 'quantization'; some are more suitable than others.

Let μ be a parameter with dimensions of mass and proportional to the Rindler acceleration. The commutative Rindler frame is given by $\tilde{\theta}^0 = \mu\tilde{x}d\tilde{t}$ and $\tilde{\theta}^1 = d\tilde{x}$; the commutative Minkowski frame is $\tilde{\theta}'^0 = d\tilde{t}'$ and $\tilde{\theta}'^1 = d\tilde{x}'$. The local Lorentz rotation from the former to the latter is defined by

$$\tilde{\theta}'^a = \tilde{\Lambda}^{-1}{}^a_b\tilde{\theta}^b \tag{6.1}$$

with

$$\tilde{\Lambda} = \begin{pmatrix} \cosh\mu\tilde{t} & \sinh\mu\tilde{t} \\ \sinh\mu\tilde{t} & \cosh\mu\tilde{t} \end{pmatrix}. \tag{6.2}$$

The classical coordinate transformation from the Rindler coordinates to the Minkowski coordinates is given, for $x > 0$ by

$$\tilde{x}' = \tilde{x}\cosh\mu\tilde{t}, \qquad \tilde{t}' = \tilde{x}\sinh\mu\tilde{t}. \tag{6.3}$$

It is of course not to be confused with the rotation.

We shall first show that the Rindler frame is not a suitable frame; there are no dual momenta. If momenta p_a did exist then they would necessarily satisfy the relations

$$\begin{aligned} &[p_0, t] = (\mu x)^{-1}, \quad &&[p_0, x] = 0, \\ &[p_1, t] = 0, &&[p_1, x] = 1. \end{aligned} \tag{6.4}$$

But one easily sees that the solution is not a quadratic algebra. In fact if one set $[p_0, p_1] = L_{01}$ and $[t, x] = i\bar{k}J^{01}$, one finds from the Jacobi identities that

$$[L_{01}, t] = [p_0, [p_1, t]] - [p_1, [p_0, t]] = \mu^{-1}x^{-2}, \tag{6.5}$$

$$[L_{01}, x] = [p_0, [p_1, x]] - [p_1, [p_0, x]] = 0. \tag{6.6}$$

The L_{01} commutes with x and therefore belongs to the algebra generated by x. From the commutation with t one finds

$$i\bar{k}\mu(\frac{d}{dx}L_{01})J^{01} = -x^{-2}. \tag{6.7}$$

Similarly one has

$$i\bar{k}[p_0, J^{01}] = [[p_0, t], x] - [[p_0, x], t] = 0, \tag{6.8}$$

$$i\bar{k}[p_1, J^{01}] = [[p_1, t], x] - [[p_1, x], t] = 0 \tag{6.9}$$

from which one concludes that J^{01} is constant. We shall set $J^{01} = 1$. We deduce therefore, neglecting integration constants, that

$$L_{01} = \frac{1}{i\bar{k}\mu}x^{-1}. \tag{6.10}$$

But the duality relations (6.4) require

$$p_0 = -\frac{1}{i\bar{k}\mu}\log(\mu x), \qquad p_1 = \frac{1}{i\bar{k}}t \tag{6.11}$$

and thus one easily sees that

$$L_{01} = \frac{1}{i\bar{k}}e^{i\bar{k}\mu p_0} \tag{6.12}$$

which is not a quadratic expression in p_0 and p_1.

The expressions (6.11) for the momenta seem quite different from the corresponding commutative expressions for the derivations $\tilde{e}_i$ dual to the frame:

$$\tilde{e}_0 = (\mu\tilde{x})^{-1}\partial_0, \qquad \tilde{e}_1 = \partial_1. \tag{6.13}$$

However in both cases one obtains the same action on the generators of the algebra. In particular

$$\tilde{e}_0\tilde{t} = (\mu\tilde{x})^{-1}, \qquad [p_0, t] = (\mu x)^{-1}. \tag{6.14}$$

Here one appreciates the importance of the space-time commutation relation.

Although the momenta p_a dual to the frame which we have used do not satisfy a quadratic relation it is easy to introduce another set $\bar{p}_a$ which do. We define the new momenta by the equations

$$\bar{p}_0 = -\frac{1}{i\hbar\mu}e^{-i\hbar\mu p_0}, \qquad \bar{p}_1 = p_1. \tag{6.15}$$

They obey the commutation relation

$$[\bar{p}_0, \bar{p}_1] = \frac{1}{i\hbar}. \tag{6.16}$$

From (6.11) one see also that $\bar{p}_a$ are related to the coordinate generators by the transformations

$$x = -i\hbar\bar{p}_0, \qquad t = i\hbar\bar{p}_1. \tag{6.17}$$

The frame defined by the new momenta is given by $\theta^a = \delta^a_i \bar{d}x^i$; it is a Minkowski-like frame in Rindler coordinates. We put here a bar on the differential to emphasize that the calculus is different. In spite of the apparent nonlocality in the transformation (6.15) the action of both $\bar{p}_0$ and p_0

$$[p_0, f] \to (\mu\tilde{x})^{-1}\tilde{\partial}_0 f, \qquad [\bar{p}_0, f] \to \tilde{\partial}_0 f \tag{6.18}$$

is local.

We now compare the differential calculi defined by two different frames, Rindler and Minkowski, and related by a noncommutative frame rotation of the type (6.1). Both frames can be used to define a differential calculus; each differential calculus has at most one basis as frame. Let $\tilde{\theta}^a$ be a global moving frame for some 2-dimensional commutative geometry and let $\{\tilde{\theta}'^a\}$ be the set of all moving frames $\tilde{\theta}'^a$ such that

$$\tilde{\theta}'^a = \tilde{\Lambda}^{-1}{}^a_b\tilde{\theta}^b \tag{6.19}$$

for some local Lorentz rotation $\tilde{\Lambda}$. The set of noncommutative versions will be described then each by a frame θ'^a which we shall suppose related by

$$\theta'^a = \Lambda^{-1}{}^a_b\theta^b \tag{6.20}$$

for the corresponding 'noncommutative' local Lorentz rotation. In the special cases we have been considering one can restrict the matrices Λ to the subset the elements of which depend on but one generator so they are well-defined. In more general situations the definition would require elaboration. It is clear that if $[f, \theta'^a] = 0$ then

$$[f, \theta^a] = [f, \Lambda^a_b]\Lambda^{-1}{}^b_c\theta^c. \tag{6.21}$$

This can also be written as a rule

$$\theta^a f = \Lambda^a_b f \Lambda^{-1b}{}_c \theta^c \tag{6.22}$$

for relating the left- and right-module structures. In general then each local rotation defines a different calculus. The equivalence class of (commutative) moving frames gives rise to a set of inequivalent frames which have the same classical limit. If one wishes to consider one calculus, defined, say, by the condition $[f, \theta^a] = 0$ then each of the bases θ'^a satisfies the relation

$$\theta'^a f = \Lambda^{-1a}{}_b f \Lambda^b_c \theta'^c. \tag{6.23}$$

That is, θ'^a is not the frame for the same calculus unless Λ is a global rotation (a constant matrix).

Note that the Rindler metric can be considered equivalent to the 2-dim Kasner metric. The Kasner metric in dimension-4, for a special value of the parameters, is flat and a moving frame can be chosen which for these values become the ordinary flat frame:

$$\tilde\theta^0 = d\tilde t, \quad \tilde\theta^1 = d\tilde x - \tilde t^{-1}\tilde x d\tilde t, \quad \theta^2 = d\tilde y, \quad \theta^3 = d\tilde z. \tag{6.24}$$

We refer here to $\{\tilde\theta^0, \tilde\theta^1\}$ as the 2-dim Kasner moving frame. By a change of variables

$$\tilde x \to \tilde t, \qquad \tilde t \to \tilde t^{-1}\tilde x. \tag{6.25}$$

the Rindler frame can be brought to this form.

7 Kasner: noncommutative corrections

We shall here study the noncommutative corrections of the Kasner metric as defined in (6.24). For convenience instead of $\tilde x$ and $\tilde t$ we choose as classical variables $\tilde t$ and

$$\tilde\phi = \tilde t^{-1}\tilde x. \tag{7.1}$$

As we have already learned, not all moving frames attached to a metric are suitable for quantization; in this case the appropriate differential calculus is that determined by the flat Minkowski frame. The classical frame rotation from the Minkowski moving frame $\tilde\theta^a$ to the Kasner moving frame $\tilde\eta^a$ is given by

$$\begin{aligned} \tilde\eta^0 &= \cosh\tilde\phi\,\tilde\theta^0 - \sinh\tilde\phi\,\tilde\theta^1, \\ \tilde\eta^1 &= -\sinh\tilde\phi\,\tilde\theta^0 + \cosh\tilde\phi\,\tilde\theta^1. \end{aligned} \tag{7.2}$$

The relation between the two coordinate systems is given by

$$\tilde x' = \tilde t \sinh\tilde\phi, \qquad \tilde t' = \tilde t\cosh\tilde\phi. \tag{7.3}$$

It follows that $\tilde t^2 = \tilde t'^2 - \tilde x'^2$; the origin of the Kasner time coordinate, exactly at the flat-space values of the parameters and because of the singular nature of the transformation, becomes a null surface.

In the noncommutative case we choose the symmetric ordering; therefore the change of generators (7.3) becomes

$$\begin{aligned} x' &= t\sinh\phi - \tfrac12[t, \sinh\phi], \\ t' &= t\cosh\phi - \tfrac12[t, \cosh\phi]. \end{aligned} \tag{7.4}$$

We normalize the Minkowski coordinates so that the commutator is given by $[t', x'] = i\bar{k}$. One finds that the corresponding Kasner commutator is

$$[t, x] = i\bar{k}(1 + o((i\bar{k})^2)). \tag{7.5}$$

We shall use rather the form

$$[t, \phi] = i\bar{k}t^{-1}(1 + o((i\bar{k})^2)). \tag{7.6}$$

The frame is given by

$$\theta^0 = dt', \qquad \theta^1 = dx'. \tag{7.7}$$

We can rewrite it in terms of t and x using the change of variables. It is of interest however to express also the calculus in terms of the Kasner moving frame; as we have already noticed in Equation (6.21) it is not a noncommutative frame since the frame rotation is local. We designate it therefore η^a and define as

$$\begin{aligned} \eta^0 &= \cosh\phi\,\theta^0 - \sinh\phi\,\theta^1, \\ \eta^1 &= -\sinh\phi\,\theta^0 + \cosh\phi\,\theta^1. \end{aligned} \tag{7.8}$$

Since from $[\phi, \theta^a] = 0$ we obtain $[\phi, \eta^a] = 0$, the equation (7.8) can be easily inverted. To complete the definition of the differential calculus we need the commutation relations $[t, \eta^a]$. These can only be calculated perturbatively. To lowest order one finds

$$[t, \eta^0] = -i\bar{k}t^{-1}\eta^1 + \tfrac{3}{2}(i\bar{k})^2 t^{-3}\eta^0, \tag{7.9}$$

$$[t, \eta^1] = -i\bar{k}t^{-1}\eta^0 + \tfrac{3}{2}(i\bar{k})^2 t^{-3}\eta^1. \tag{7.10}$$

To the same order, the transformation (7.4) of the generators reads

$$\begin{aligned} x' &= \sinh\phi\, t + \tfrac{1}{2}i\bar{k}\cosh\phi\, t^{-1} + \tfrac{1}{4}(i\bar{k})^2\sinh\phi\, t^{-3}, \\ t' &= \cosh\phi\, t + \tfrac{1}{2}i\bar{k}\sinh\phi\, t^{-1} + \tfrac{1}{4}(i\bar{k})^2\cosh\phi\, t^{-3}. \end{aligned} \tag{7.11}$$

These commutation relations determine a noncommutative geometry which is a natural extension of the flat Kasner geometry.

8 Fuzzy donut: momentum relations

Let us examine now further properties of the module structure defined by the differential (4.3). The exterior product is a map from the tensor product of two copies of the module of 1-forms into the module of 2-forms. We shall identify the latter as a subset of the former and write the product as

$$\theta^a\theta^b = P^{ab}{}_{cd}\theta^c \otimes \theta^d. \tag{8.1}$$

The $P^{ab}{}_{cd}$ are complex numbers which satisfy the projector condition and a hermiticity condition [6]. The basis 1-forms anticommute for $P^{ab}{}_{cd} = \frac{1}{2}(\delta^a_c\delta^b_d - \delta^b_c\delta^a_d)$. The exterior derivative of θ^a is a 2-form, so it can be written as

$$d\theta^a = -\frac{1}{2}C^a{}_{bc}\,\theta^b\theta^c. \tag{8.2}$$

The $C^a{}_{bc}$ are called the structure elements. They can be chosen to satisfy $C^a{}_{bc} = C^a{}_{de}P^{de}{}_{bc}$.

Impose now the condition $d^2 = 0$. It gives

$$0 = d(df) = d(-[\theta, f]) = -\left[d\theta + \theta^2, f\right], \tag{8.3}$$

so it implies that $d\theta + \theta^2$ commutes with all elements of the algebra. Since $d\theta + \theta^2$ is a 2-form, in the frame basis it can be written as

$$d\theta + \theta^2 = -\frac{1}{2} K_{ab}\theta^a\theta^b \tag{8.4}$$

where the elements K_{ab} are complex numbers. One can impose $K_{ab}P^{ab}{}_{cd} = K_{cd}$. A straightforward calculation shows that

$$\begin{aligned} d\theta &= -dp_a\theta^a - p_a d\theta^a = [p_b, p_a]\,\theta^b\theta^a + \frac{1}{2}p_a C^a{}_{bc}\theta^b\theta^c, \\ \theta^2 &= p_a p_b \theta^a\theta^b, \end{aligned} \tag{8.5}$$

and hence (8.4) reduces to

$$(p_c p_b + \frac{1}{2}C^a{}_{bc}p_a + \frac{1}{2}K_{bc})\theta^b\theta^c = 0. \tag{8.6}$$

This can be written as

$$2p_b p_a P^{ab}{}_{cd} + p_a C^a{}_{cd} + K_{cd} = 0. \tag{8.7}$$

The relation $d(f\theta^a - \theta^a f) = 0$ written in terms of the momenta gives further restrictions. It reads

$$[p_b\delta^a_c + p_c\delta^a_b + \frac{1}{2}C^a{}_{bc}, f]\theta^b\theta^c = 0 \tag{8.8}$$

which means

$$(C^a{}_{bc} + 2p_b\delta^a_c + 2p_c\delta^a_b - F^a{}_{bc})\,\theta^b\theta^c = 0, \tag{8.9}$$

where $F^a{}_{bc}$ are complex numbers. Thus the structure elements, defined in Equation (8.2) are linear in the momenta. It follows immediately that

$$e_a C^a{}_{bc} = 0. \tag{8.10}$$

This relation must be also satisfied in the commutative limit and constitutes a constraint on the frame. The example of Section 6 shows that this condition is not necessarily sufficient. A frame has four degrees of freedom in two dimensions. The constraint subtracts one therefrom. On the other hand having chosen a calculus, the choice of frame is equivalent to a gauge condition. This can be made more transparent if the momenta exist in which case the gauge condition can be expressed as the condition (8.10). The commutation relation (4.5) can be thought of also as a gauge condition since it is necessary for the existence of the momenta; there remain hence $4 - 1 - 1 = 2$ degrees of freedom. Combining (8.7) and (8.9) we obtain the relation

$$2p_c p_d P^{cd}{}_{ab} - p_c F^c{}_{ab} - K_{ab} = 0. \tag{8.11}$$

The coefficients in (8.11) are complex numbers. We see that the momentum generators p_a satisfy a quadratic relation.

One can readily find the conjugate momenta for a family of 2-dim metrics with one Killing vector. We shall exhibit all possible choices which yield differential calculi based on inner derivations. As frame we choose

$$\theta^0 = f(x)dt, \quad f > 0, \qquad \theta^1 = dx. \tag{8.12}$$

and we suppose that $J^{01} = J^{01}(x)$. The frame relations can be written as

$$\begin{aligned} dx\, x &= x\, dx, \qquad & dx\, t &= t\, dx, \\ dt\, x &= x\, dt, \qquad & dt\, t &= (t + i\bar{k} F) dt. \end{aligned} \tag{8.13}$$

We have set, for convenience

$$F = J^{01} \frac{d}{dx} \log f. \tag{8.14}$$

The differential structure of the algebra can be written as

$$(dx)^2 = 0, \quad dx\, dt = -dt\, dx, \quad (dt)^2 = -\tfrac{1}{2} i\bar{k} F' dx\, dt \tag{8.15}$$

or as the relations

$$(\theta^1)^2 = 0, \qquad \theta^0\theta^1 = -\theta^1\theta^0, \tag{8.16}$$

$$(\theta^0)^2 = \tfrac{1}{2} i\bar{k} f F' \theta^0\theta^1 = 2i\epsilon\theta^0\theta^1 \tag{8.17}$$

with

$$\epsilon = 4\bar{k} f F'. \tag{8.18}$$

It follows from the frame properties that ϵ is a constant.

Suppose now that the dual momenta exist. The duality relations are

$$\begin{aligned} [p_0, t] &= f^{-1}, \quad & [p_0, x] &= 0, \\ [p_1, t] &= 0, & [p_1, x] &= 1. \end{aligned} \tag{8.19}$$

These relations allow us to identify p_1 with the partial derivative with respect to x. If $\phi = \phi(x)$ then

$$[p_1, \phi] = [p_1, x]\partial_x \phi = \partial_x \phi. \tag{8.20}$$

On the other hand, for $\phi = \phi(t, x)$ we can write to first order

$$[p_0, \phi] = [p_0, t]\partial_t \phi = f^{-1}\partial_t \phi. \tag{8.21}$$

If we denote as before $[p_0, p_1] = L_{01}$, the Jacobi identities imply the relations

$$\begin{aligned} [p_0, J^{01}] &= 0, \qquad & [p_1, J^{01}] &= 0, \\ [t, L_{01}] &= -f' f^{-2}, & [x, L_{01}] &= 0. \end{aligned} \tag{8.22}$$

One can conclude again that J^{01} is constant and also that L_{01} is a function of x alone. We set $J^{01} = 1$. It follows that, neglecting the integration constants, the 'Fourier transformation' between the position and momentum generators is given by

$$p_0 = -\frac{1}{i\bar{k}} \int f^{-1}, \qquad p_1 = -\frac{1}{i\bar{k}}\, t. \tag{8.23}$$

Each of the pairs (t, x) and (p_0, p_1) generates the algebra.

The array $P^{ab}{}_{cd}$ we write as

$$P^{ab}{}_{cd} = \tfrac{1}{2}\delta^{[a}_c \delta^{b]}_d + i\epsilon Q^{ab}{}_{cd} \tag{8.24}$$

In dimension two, if we assume that metric depends on x that is on p_0 only, we find that

$$P^{ab}{}_{cd} p_a p_b = \frac{1}{2}[p_c, p_d] + i\epsilon Q^{00}{}_{cd} p_0^2 \tag{8.25}$$

and therefore L_{01} is given by

$$L_{01} = K_{01} + p_0 F^0{}_{01} - 2i\epsilon p_0^2 Q^{00}{}_{01}. \tag{8.26}$$

The structure elements are given by

$$C^0{}_{01} = F^0{}_{01} - 4i\epsilon p_0 Q^{00}{}_{01}. \tag{8.27}$$

Symmetry and reality of the product imply that $Q^{ab}{}_{cd}$ has non-vanishing elements:

$$Q^{10}{}_{00} = -Q^{01}{}_{00} = 1, \qquad Q^{00}{}_{01} = -Q^{00}{}_{10} = 1. \tag{8.28}$$

We set also

$$K_{01} = \frac{1}{i\hbar J^{01}} = \frac{1}{i\hbar}, \qquad F^0{}_{01} = -ib\mu, \tag{8.29}$$

while $C^0{}_{10}$ is determined by the constraint

$$C^0{}_{ab} P^{ab}{}_{01} = C^0{}_{01}, \qquad C^0{}_{01} + C^0{}_{10} = -2i\epsilon C^0{}_{00}. \tag{8.30}$$

We have then finally the expressions

$$L_{01} = (i\hbar)^{-1}(1 - b\mu^{-1}(i\epsilon p_0) - 2\mu^{-2}(i\epsilon p_0)^2), \tag{8.31}$$

$$C^0{}_{01} = -ib\mu - 4i\epsilon p_0, \tag{8.32}$$

and a differential equation

$$-i\epsilon \frac{dp_0}{dx} = \mu^2 - i\epsilon b\mu p_0 - 2(i\epsilon p_0)^2 \tag{8.33}$$

for p_0. There are three cases to be considered.

The simplest is the case with $\mu^2 \to \infty$. The equation reduces to

$$-i\hbar \frac{dp_0}{dx} = 1. \tag{8.34}$$

One finds the relations

$$i\hbar p_0 = -x, \qquad f(x) = 1. \tag{8.35}$$

This is noncommutative Minkowski space.

An equally degenerate case is the case with $\mu^2 \to \infty$ and with $\epsilon b = c\mu$. Equation (8.33) can be written in the form

$$-i\hbar \frac{dp_0}{dx} = 1 - icp_0. \tag{8.36}$$

One finds the solution

$$ip_0 = c^{-1}(e^{-\hbar^{-1}cx} - 1), \qquad f(x) = e^{\hbar^{-1}cx}. \tag{8.37}$$

The change of variables

$$t' = 2t, \qquad \mu x' = 2c^{-1}(e^{-cx} - 1), \tag{8.38}$$

transforms the algebra into the algebra of de Sitter space analyzed in Section 4.

The case which interests us the most here is that with μ finite. With $b = 0$ the equation becomes

$$-i\epsilon \frac{dp_0}{dx} = \mu^2 - 2(i\epsilon p_0)^2. \tag{8.39}$$

If we introduce the notation

$$\beta^2 = 2\mu^2 > 0 \tag{8.40}$$

the equation becomes

$$\frac{1}{\beta}\frac{d}{dx}\left(-2i\epsilon\beta^{-1}p_0\right) = 1 - \left(-2i\epsilon\beta^{-1}p_0\right)^2. \tag{8.41}$$

The solution is given by

$$i\kern-.3em{}^{-}\!\!k p_0 = -\beta^{-1}\tanh(\beta x), \qquad f(x) = \cosh^2(\beta x). \tag{8.42}$$

The function F is

$$F = -2i\beta^2 \bar{k} p_0 = 2\beta\tanh(\beta x). \tag{8.43}$$

We find therefore the identity

$$F' + F^2 = f^{-1}f'' = 2\beta^2(1 + \tanh^2(\beta x)). \tag{8.44}$$

The frame (8.12) is given by

$$\theta^0 = \cosh^2(\beta x)dt = \tfrac{1}{2}(1 + \cosh(2\beta x))dt, \qquad \theta^1 = dx. \tag{8.45}$$

Frames of similar type have appeared [16, 17, 18] in 2-dimensional dilaton gravity theories. In the commutative limit the connection and the curvature which correspond to this frame are

$$\tilde{\omega}^0{}_1 = \tilde{\omega}^1{}_0 = F\tilde{\theta}^0, \qquad \tilde{\Omega}^0{}_1 = \tilde{\Omega}^1{}_0 = -(F' + F^2)\theta^0\theta^1 = -f^{-1}f''\tilde{\theta}^0\tilde{\theta}^1. \tag{8.46}$$

The solution is a completely regular manifold of Minkowski signature which has the Rindler metric as singular limit. In the limit $\beta \to 0$

$$i\bar{k}p_0 = -x, \qquad f = 1, \tag{8.47}$$

and one finds Minkowski space. In 'tortoise' coordinate x^*,

$$x^* = \int \frac{dx}{f(x)} \tag{8.48}$$

the frame is given by

$$\theta^0 = \frac{1}{1 - x^{*2}}\,dt, \qquad \theta^1 = \frac{1}{1 - x^{*2}}\,dx^*. \tag{8.49}$$

From (8.23) we see that $x^* = -i\bar{k}p_0$.

Under a Wick rotation

$$u = 2i\beta x, \qquad v = t \tag{8.50}$$

the frame (8.12) becomes

$$\theta^0 = \tfrac{1}{2}(1 + \cos u)dv, \qquad \theta^1 = \frac{1}{2i\beta}\,du \tag{8.51}$$

and the line element in the commutative limit has the form

$$ds^2 = \tfrac{1}{4}(1 + \cos\tilde{u})^2 d\tilde{v}^2 + \tfrac{1}{4}\beta^{-2} d\tilde{u}^2. \tag{8.52}$$

This is the surface of the torus embedded in $\mathbb{R}^3$:

$$\tilde{x} = \tfrac{1}{2}(1 + \cos\tilde{u})\cos\tilde{v}, \quad \tilde{y} = \tfrac{1}{2}(1 + \cos\tilde{u})\sin\tilde{v}, \quad \tilde{z} = \tfrac{1}{2}\beta^{-1}\sin\tilde{u}, \tag{8.53}$$

and for this reason we call this metric the 'fuzzy donut'. It is a singular axially-symmetric surface of Gaussian curvature

$$\tilde{K} = 2\beta^2(1 - \tan^2\tfrac{1}{2}\tilde{u}). \tag{8.54}$$

The donut is defined by the coordinate range $0 \leq u \leq 2\pi$, $0 \leq v \leq 2\pi$, with a singularity at the point $u = \pi$. In spite of the singularity, the Euler characteristic is given by

$$e[\c{A}] = \frac{1}{4\pi}\epsilon_{ab}\int \tilde{\Omega}^{ab} = -\frac{1}{2\pi}\int \tilde{\Omega}^0{}_1 = -\frac{1}{2\pi}\int d\tilde{\omega}^0{}_1 = 0 \tag{8.55}$$

as it should be. If we suppose the same domain in the Wick rotated real-t region, then

$$0 \leq x \leq \beta^{-1}\pi, \qquad 0 \leq t \leq 2\pi. \tag{8.56}$$

As $\beta \to \infty$ the donut becomes more and more squashed, and this domain becomes an elementary domain in the limiting Minkowski space.

9 Noncommutative differential geometry

We have presented several noncommutative 'blurings' of classical geometries, all of which are of dimension two. We have concentrated our attention on the new aspects of the noncommutative theory, especially the plethora of differential calculi and the relation of the geometry to the symplectic structures. We have not, in fact, introduced the metric, the connection or the curvature on the noncommutative space. This can be done by taking the commutative limit and using the definition of a metric in terms of the frame. It can also be done [3] before the limit is taken. To complete the analysis of the family of examples discussed in Section 8, we mention the linear connections, the metric and the curvature without defining them in the full rigor; for details we refer to [8]. Note that when the momenta exist the metric is given; otherwise there is a certain ambiguity which must be determined by field equations.

To define a linear connection one needs a 'flip' [19, 20],

$$\sigma(\theta^a \otimes \theta^b) = S^{ab}{}_{cd}\theta^c \otimes \theta^d, \tag{9.1}$$

which in the present notation is equivalent to a 4-index set of complex numbers $S^{ab}{}_{cd}$ which we can write as

$$S^{ab}{}_{cd} = \delta^b_c\delta^a_d + i\epsilon T^{ab}{}_{cd}. \tag{9.2}$$

The covariant derivative is given by

$$D\xi = \sigma(\xi \otimes \theta) - \theta \otimes \xi. \tag{9.3}$$

In particular

$$D\theta^a = -\omega^a{}_c \otimes \theta^c = -(S^{ab}{}_{cd} - \delta^b_c\delta^a_d)p_b\theta^c \otimes \theta^d = -i\epsilon T^{ab}{}_{cd}p_b\theta^c \otimes \theta^d, \tag{9.4}$$

so the connection-form coefficients are linear in the momenta

$$\omega^a{}_c = \omega^a{}_{bc}\theta^b = i\epsilon p_d T^{ad}{}_{bc}\theta^b. \tag{9.5}$$

On the left-hand side of the last equation is a quantity $\omega^a{}_c$ which measures the variation of the metric; on the right-hand side is the array $T^{ad}{}_{bc}$ which is directly related to the anti-commutation rules for the 1-forms, and more important the momenta p_d which define the frame. As $\bar{k} \to 0$ the right-hand side remains finite and

$$\omega^a{}_c \to \tilde{\omega}^a{}_c. \tag{9.6}$$

The identification is only valid in the weak-field approximation. The connection is torsion-free if the components satisfy the constraint

$$\omega^a{}_{ef} P^{ef}{}_{bc} = \tfrac{1}{2} C^a{}_{bc}. \tag{9.7}$$

The metric is a map

$$g:\ \Omega^1(\underset{\sim}{A}) \otimes \Omega^1(\underset{\sim}{A}) \to \underset{\sim}{A}. \tag{9.8}$$

Using the frame it is defined by

$$g(\theta^a \otimes \theta^b) = g^{ab}, \tag{9.9}$$

and bilinearity of the metric implies that g^{ab} are complex numbers. In the present formalism [3] the metric is ‘real’ if it satisfies the condition

$$\bar{g}^{ba} = S^{ab}{}_{cd} g^{cd}. \tag{9.10}$$

‘Symmetry’ of the metric can be defined either using the projection

$$P^{ab}{}_{cd} g^{cd} = 0, \tag{9.11}$$

or the flip

$$S^{ab}{}_{cd} g^{cd} = c g^{ab}. \tag{9.12}$$

We usually take the frame to be orthonormal in the commutative limit, therefore one can write the metric as

$$g^{ab} = \eta^{ab} + i\epsilon h^{ab}. \tag{9.13}$$

In the linear approximation, the condition of the reality of the metric becomes

$$h^{ab} + \bar{h}^{ab} = -T^{ba}{}_{cd}\eta^{cd}. \tag{9.14}$$

The connection is metric if

$$\omega^a{}_{bc} g^{cd} + \omega^d{}_{ce} S^{ac}{}_{bf} g^{fe} = 0, \tag{9.15}$$

or linearized,

$$T^{(ac}{}_d{}^{b)} = 0. \tag{9.16}$$

In our 2-dim model the frame is of the form

$$\theta^0 = f(x)dt, \qquad \theta^1 = dx. \tag{9.17}$$

The torsion-free metric-compatible connection and the curvature are classically given by the expressions (8.46). From these expressions we see that the geometry is flat only

if $f(x)$ is linear in x. We recall that $\epsilon = \bar{k}\mu^2$. To first order the fuzzy calculus differs from the commutative limit in the two relations

$$\begin{aligned} \theta^0\theta^1 &= P^{01}{}_{ab}\theta^a\theta^b = \tfrac{1}{2}\theta^{[0}\theta^{1]} + i\epsilon Q^{01}{}_{00}(\theta^0)^2 \\ &= \tfrac{1}{2}\theta^{[0}\theta^{1]} - i\epsilon q(\theta^0)^2 \end{aligned} \tag{9.18}$$

$$(\theta^0)^2 = P^{00}{}_{ab}\theta^a\theta^b = i\epsilon Q^{00}{}_{01}\theta^{[0}\theta^{1]} = i\epsilon q\theta^{[0}\theta^{1]}. \tag{9.19}$$

These can be better written as

$$\theta^{(0}\theta^{1)} = -2i\epsilon q(\theta^0)^2, \quad (\theta^0)^2 = i\epsilon q\theta^{[0}\theta^{1]}, \tag{9.20}$$

and to first order reduce to

$$\theta^{(0}\theta^{1)} = 0, \quad (\theta^0)^2 = 2i\epsilon q\theta^0\theta^1. \tag{9.21}$$

The quantity q which we have introduced in (9.18-9.21) is a constant, $q = 0$ in the cases of flat and de Sitter noncommutative spaces and $q = 1$ in the fuzzy donut case. We will restrict our considerations to the latter.

The differentials of the frame are given by

$$d\theta^0 = -C^0{}_{01}\theta^0\theta^1, \qquad d\theta^1 = 0, \tag{9.22}$$

with

$$C^0{}_{01} = -4i\epsilon p_0 Q^{00}{}_{01} = -4i\epsilon p_0. \tag{9.23}$$

The only non-vanishing components of the connection are

$$\omega^0{}_1 = \omega^1{}_0 = -4i\epsilon p_0\theta^0 = F\theta^0, \tag{9.24}$$

and from (9.5) we find

$$T^{00}{}_{01} = T^{10}{}_{00} = -4. \tag{9.25}$$

To first order the condition that the torsion vanish is the equation (9.7); it is satisfied by the values we obtain. The curvature 2-form has components

$$\Omega^0{}_1 = -(F' + F^2)\,\theta^0\theta^1 \tag{9.26}$$

$$\Omega^0{}_0 = \Omega^0{}_0 = 2i\epsilon F^2\theta^0\theta^1. \tag{9.27}$$

Therefore to lowest order from (8.44) we find the Gaussian curvature

$$\Omega^0{}_1 = -2\beta^2(1 + \tanh^2(\beta x))\theta^0\theta^1. \tag{9.28}$$

We must define a 'real', 'symmetric' metric. There are in principle four possible ways to define it depending on which of two possible ways one chooses to define symmetry, and whether or not one includes a twist in the extension of the metric to the tensor product. In all cases the torsion-free condition yields the relation

$$T^{abcd} = 2(Q_-^{bcda} + Q_-^{bdca} + Q_-^{abdc}), \tag{9.29}$$

and the reality of the metric

$$h^{ab} + \bar{h}^{ab} = -T^{ba}{}_{cd}\eta^{cd}, \tag{9.30}$$

both in the linear approximation. Here we denote $Q^{ab}_{-\,cd} = \frac{1}{2}Q^{ab}{}_{[cd]}$, $Q^{ab}_{+\,cd} = \frac{1}{2}Q^{ab}{}_{(cd)}$. The projector $P^{ab}{}_{cd}$ is hermitean if

$$Q_+^{abcd} = \pm Q_-^{cdab}, \tag{9.31}$$

with plus in the case of no twist and minus with twist. If one use the flip to define symmetry, then for some γ the linearized perturbation must satisfy

$$h^{[ab]} = T^{ab}{}_{cd}\eta^{cd} - \gamma\eta^{ab} \tag{9.32}$$

if the metric is to be symmetric. If one use the product to define symmetry then

$$h^{[ab]} = -2Q_+^{abcd}\eta_{cd}. \tag{9.33}$$

In the present example the only consistent choice is the following

$$h^{[ab]} = -2Q_+^{abcd}\eta_{cd} = -2Q_-^{cdab}\eta_{cd}. \tag{9.34}$$

Thus for the symmetric and real metric we obtain

$$g^{ab} = \eta^{ab} + i\epsilon h^{ab}, \qquad h^{ab} = 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{9.35}$$

The η^{ab} here is the matrix of components of the canonical Minkowski metric; to it can be added an antisymmetric real matrix which is not fixed:

$$\eta^{ab} \mapsto \eta^{ab} + \epsilon\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}. \tag{9.36}$$

This ambiguity exists already at the classical level.

10 Higher-order effects

To find the second order corrections to our system, we write the 4-index tensors as matrices ordering the indices $(01, 10, 11, 00)$. Let P_0 and S_0 be respectively the canonical projector and the flip

$$P_0 = \begin{pmatrix} 1/2 & -1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad S_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10.1}$$

The projector constraints are, in matrix notation,

$$P^2 = P, \qquad \bar{P}\hat{P} = \hat{P} \tag{10.2}$$

where $\hat{A}^{ab}{}_{cd} = A^{ba}{}_{cd} = (S_0 A)^{ab}{}_{cd}$. To lowest order these conditions become

$$\bar{Q} = Q, \qquad \hat{Q}_- = Q_-. \tag{10.3}$$

The twist constraints are

$$\hat{\bar{S}}\hat{S} = 1, \tag{10.4}$$

$$\hat{S}P + \bar{P}\hat{P} = 0, \tag{10.5}$$

$$SP + P = 0. \tag{10.6}$$

The last two identities are equivalent if

$$\bar{P}\hat{P} = \hat{P}. \tag{10.7}$$

This condition was already imposed in (10.2).

One can easily check that the first order solution of the previous section which we write as $P = P_0 + i\epsilon Q$, $S = S_0 + i\epsilon T$, is given by

$$Q = Q_- + Q_+, \qquad Q_- = \begin{pmatrix} 0 & 0 \\ \tau & 0 \end{pmatrix}, \qquad Q_+ = \begin{pmatrix} 0 & -\tau^* \\ 0 & 0 \end{pmatrix}, \tag{10.8}$$

and

$$T = -2 \begin{pmatrix} 0 & \tau\tau^* \\ \tau\tau^*\sigma_1 & 0 \end{pmatrix}. \tag{10.9}$$

We introduced the matrix τ and its transpose τ^*:

$$\tau = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \qquad \tau\tau^* = 1 - \sigma_3, \quad \tau^*\tau = 1 - \sigma_1. \tag{10.10}$$

The constraints (10.2), (10.5-10.6) can be solved to second order using inner automorphisms of the matrix algebra. Denote

$$P = P_0 + i\epsilon Q + (i\epsilon)^2 Q_2, \tag{10.11}$$

$$S = S_0 + i\epsilon T + (i\epsilon)^2 T_2 \tag{10.12}$$

and introduce the automorphism $P = W^{-1} P_0 W$, where W is an arbitrary nonsingular 4×4 matrix with inverse W^{-1}. We see immediately that $P^2 = P$. To satisfy the second condition of (10.2) on P it is sufficient to require that

$$\bar{W} S_0 = S_0 W, \tag{10.13}$$

and to recall $\hat{S} = S_0 S$. Let $W = \exp(i\epsilon B)$. To second order

$$P = P_0 + i\epsilon[P_0, B] + \tfrac{1}{2}(i\epsilon)^2[[P_0, B], B], \tag{10.14}$$

and the two expansions coincide if

$$Q = [P_0, B], \qquad Q_2 = \tfrac{1}{2}[[P_0, B], B] = \tfrac{1}{2}[Q, B]. \tag{10.15}$$

It is easy to see that an appropriate solution is

$$B = \begin{pmatrix} 0 & -\tau^* \\ -\tau & 0 \end{pmatrix}. \tag{10.16}$$

One can also check

$$P_0 B = Q_+, \quad B P_0 = -Q_-, \quad S_0 B = -B S_0 = -Q. \tag{10.17}$$

The (10.13) becomes the condition $\bar{B} S_0 = -S_0 B$, which in turn, since B is real, is the condition that B and S_0 anticommute.

The solution for T_2 is

$$T_2 = \tfrac{1}{2} T S_0 T. \tag{10.18}$$

To check whether the twist constraints hold, introduce

$$A = \begin{pmatrix} 0 & \tau\tau^* \\ \tau\tau^* & 0 \end{pmatrix}, \qquad T = -2AS_0, \quad (A - B)P_0 = 0. \tag{10.19}$$

At least to second order we have

$$S = S_0 X, \quad S = Y S_0, \qquad X = \exp(i\epsilon S_0 T), \quad Y = \exp(i\epsilon T S_0). \tag{10.20}$$

The twist constraint

$$\hat{\bar{S}}\hat{S} = S_0(S_0\bar{X})S_0(S_0X) = S_0^2\bar{X}X = 1 \tag{10.21}$$

follows. Further, consider the identity

$$\begin{aligned} &(1 - WYW)P_0 = (1 - \exp(i\epsilon B)\exp(i\epsilon T S_0)\exp(i\epsilon B)P_0 \\ &= -i\epsilon(TS_0 + 2B)P_0 + \cdots = i\epsilon(T - 2B)P_0 + \cdots \\ &= i\epsilon(T + 2Q)P_0 + \cdots = (i\epsilon)^2 H + o((i\epsilon)^3) \end{aligned} \tag{10.22}$$

with

$$H = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10.23}$$

To lowest order, therefore,

$$\begin{aligned} SP + P &= YS_0W^{-1}P_0W + P = YWS_0P_0W + P \\ &= W^{-1}(1 - WYW)P_0W = (i\epsilon)^2[H, B] + \cdots \end{aligned} \tag{10.24}$$

so all constraints are satisfied at least to second order.

The second-order metric is

$$g = \sqrt{\bar{X}} g_0. \tag{10.25}$$

To analyse the metric we write it as a 4-vector. We see then that if $\bar{g}_0 = g_0$,

$$\hat{\bar{g}} = S_0\sqrt{X}g_0 = S_0X\sqrt{\bar{X}}g_0 = Sg. \tag{10.26}$$

The metric is real. Since also

$$Pg = W^{-1}P_0W\sqrt{\bar{X}}g_0 = W^{-1}[P_0, W\sqrt{\bar{X}}]g_0 \tag{10.27}$$

the metric is symmetric to the extent that

$$[P_0, W\sqrt{\bar{X}}]g_0 = 0. \tag{10.28}$$

To first order this condition becomes

$$g = g_0 - \tfrac{1}{2}i\epsilon S_0 T g_0 == \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + 2i\epsilon \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{10.29}$$

We saw in Section 9 that this metric is compatible with the connection

$$\omega^a{}_b = S^{ac}{}_{db}\theta^d p_c + \theta\delta^a_b, \qquad \theta = -p_a\theta^a \tag{10.30}$$

to first order in the expansion parameter.

In general a connection is metric-compatible if the condition

$$\omega^i{}_{kl} g^{lj} + \omega^j{}_{ln} S^{il}{}_{km} g^{mn} = 0 \tag{10.31}$$

is satisfied. This can be written in a more familiar form if one introduce the 'covariant derivative'

$$D_i X^j = \omega^j{}_{ik} X^k \tag{10.32}$$

which is twisted:

$$D_i(X^j Y^k) = D_i X^j Y^k + S^{jl}{}_{im} X^m D_l Y^k. \tag{10.33}$$

Condition (10.31) becomes then

$$D_i g^{jk} = 0. \tag{10.34}$$

If $F^i{}_{jk} = 0$ then one can also express the condition as

$$S^{im}{}_{ln} g^{np} S^{jk}{}_{mp} = g^{ij} \delta^k_l. \tag{10.35}$$

We have not succeeded in finding a connection which is metric-compatible and torsion-free to second order; there are, however, solutions with torsion which are metric-compatible.

11 Conclusions

Several models have been found which illustrate a close relation between noncommutative geometry in its 'frame-formalism' version and classical gravity. Heuristically, but incorrectly, one can formulate the relation by stating that gravity is the field which appears when one quantizes the coordinates much as the Schrödinger wave function encodes the uncertainty resulting from the quantization of phase space.

The first and simplest of these is the fuzzy-sphere which is a noncommutative geometry which can be identified with the 2-dimensional (euclidean) 'gravity' of the 2-sphere. The algebra in this case is an $n \times n$ matrix algebra; if the sphere has radius r then the parameter r/n can be interpreted as a lattice length. With the identification this model illustrates how gravity can act as an ultraviolet cutoff, a regularization which is very similar to the 'point splitting' technique which has been used when quantizing a field in classical curved backgrounds. It can also be compared with the screening of electrons in plasma physics, which gives rise to a Debye length proportional to the inverse of the electron-number density n. The analogous 'screening' of an electron by virtual electron-positron pairs is responsible for the reduction of the electron self-energy from a linear to logarithmic dependence on the classical electron radius. Other models have been found which illustrate the identification including an infinite series in all dimensions.

In the present paper yet another model is given, one which although representing a classical manifold of dimension 2 is of interest because the classical 'gravity' which arises has a varying Gaussian curvature. The authors will leave to a subsequent article the delicate task of explaining exactly which property of the metric makes it 'quantizable'. This geometry could furnish a convenient model to study noncommutative effects, for example in the colliding-D-brane description of the Big-Bang proposed by Turok & Steinhardt [26]. The 2-space describing the time evolution of the separation of the branes has been shown to be conveniently described using Rindler coordinates. One

can blur this geometry by using the metric and connection described here. The flat geometry would have to be replaced by the one given in this section; in the limit $q \to 0$ it would become flat.

The donut example is of importance in that is is the first explicit construction of an algebra and differential calculus which is singularity-free in the Minkowksi-signature domain and which has a non-constant curvature. There are two aspects of this problem. To construct a classical manifold from a differential calculus is relatively simple once one has constructed the frame. One takes formally the limit and uses the so constructed moving frame to define the metric. This is contained in the upper right of the following little diagram

$$\begin{array}{ccc} \begin{matrix}\text{Fuzzy}\\ \text{Frame}\end{matrix} & \longrightarrow & \begin{matrix}\text{Classical}\\ \text{Frame}\end{matrix} \\ \Big\downarrow & & \Big\downarrow \\ \begin{matrix}\textit{Fuzzy}\\ \textit{Geometry}\end{matrix} & \longrightarrow & \begin{matrix}\text{Classical}\\ \text{Geometry}\end{matrix} \end{array} \tag{11.1}$$

More difficult is the construction of a 'fuzzy geometry' which would fill in the lower left of the diagram and would be such that the classical geometry is a limit thereof. But this step is very important since it gives an extension of the right-hand side into what could eventually be a domain of quantum geometry. It is the box in the to-be-constructed lower left corner where possibly one can find an interesting extension of the metric containing correction terms which describe the noncommutative structure.

We have not succeeded however to completely extend this geometry to all orders in the noncommutativity parameter $i\epsilon$. This will be considered in a subsequent article. There is evidence that the extension will involve a non-vanishing value of the torsion 2-form. The metric is extended into the noncommutative domain so as to maintain such formal properties as reality and symmetry. The interpretation however as a length requires more attention when the 'coordinates' do not commute.

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